Complex Derivatives and Cauchy-Riemann Equations

Complex Derivative. Let $z = x + iy$ be the coordinate of the complex plane $\mathbb{C}$ and let $f(z)$ be a complex-valued function on some open neighborhood U of some point $c = a + ib$ of $\mathbf{C}$ (with $a, b \in \mathbb{R}$). The *complex derivative* of f at c, denoted by $f'(c)$, is defined as the limit of the difference quotient

$$\frac{f(z) - f(c)}{z - c}$$

as $z \longrightarrow c$ if such a limit exists. More precisely, $f'(c)$ is defined as the complex number L satisfying the property that for any given $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$\left| \frac{f(z) - f(c)}{z - c} - L \right| < \varepsilon$$

whenever $0 < |z - c| < \delta$. At this point, *formally* the notion of a complex derivative is completely analogous to that of the derivative of a *real-valued* function of a *single real variable*. The difference between the two kinds of derivatives, the complex derivative and the derivative of a real-valued function of a single real variable manifests itself when in the case of the complex derivative one restricts the complex variable to the horizontal line and then to the vertical line in the limit process. These two restrictions give rise to the Cauchy-Riemann equation.

Cauchy-Riemann Equation. If $f'(c)$ exists, by specializing to the vertical line $y = b$ when the limit of the difference quotient is taken we get

$$f'(c) = \frac{\partial f}{\partial x}(c).$$

Likewise, if $f'(c)$ exists, by specializing to the horizontal line $x = a$ when the limit of the difference quotient is taken we get

$$f'(c) = \frac{1}{i}\frac{\partial f}{\partial y}(c).$$

Note that there is a factor $\frac{1}{i}$ in the second expression, because $z - c = i(y - b)$ when $x = a$ whereas $z - c = x - a$ when $y = b$. Hence we have the Cauchy-Riemann equation

$$\frac{\partial f}{\partial x}(c) = \frac{1}{i}\frac{\partial f}{\partial y}(c)$$

when $f'(c)$ exists. When we write $f = u + iv$ with u and v being the real and imaginary part of f respectively, we can rewrite the Cauchy-Riemann equation as

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$$

by equating the real and imaginary parts of

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right).$$