Golden Mean Formulae

This document is a collection of definitions, formulas and examples concerned with the Golden Mean number ϕ (Phi), the Fibonacci and the Lucas numbers. The Fibonacci and the Lucas numbers are defined for integer, real and complex values.

There are decades of the formulas concerned with the Fibonacci and with the Lucas numbers which are not stated in this document. The main stress here is given on the selection of the formulas, which involve the powers of the Golden Mean with the Fibonacci and the Lucas numbers.

Definition of the ϕ as a number

$$\phi = (\sqrt{5}+1)/2 \qquad (1)$$

derived equations:

$$\phi^{-1} = (\sqrt{5}-1)/2 \qquad (1a)$$

$$-\phi^{-1} = (1-\sqrt{5})/2 \qquad (1b)$$

$$\sqrt{5} = 2\phi-1 \qquad (1c)$$

Every number or expression with $\sqrt{5}$ includes also a number ϕ and vice versa.

Table of basic arithmetic operations between ϕ, ϕ^{-1}, 1 and $\sqrt{5}$

+	ϕ	ϕ^{-1}	1	$\sqrt{5}$
ϕ	2ϕ	$\sqrt{5}$	ϕ^2	$3\phi-1$
ϕ^{-1}	$\sqrt{5}$	$2\phi^{-1}$	ϕ	$3\phi-2$
1	ϕ^2	ϕ	2	2ϕ
$\sqrt{5}$	$3\phi-1$	$3\phi-2$	2ϕ	$2\sqrt{5}$

*	ϕ	ϕ^{-1}	$\sqrt{5}$
ϕ	ϕ^2	1	$\sqrt{5}\phi$
ϕ^{-1}	1	ϕ^{-2}	$\sqrt{5}\phi^{-1}$
$\sqrt{5}$	$\sqrt{5}\phi$	$\sqrt{5}\phi^{-1}$	5

-	ϕ	ϕ^{-1}	1	$\sqrt{5}$
ϕ	0	1	ϕ^{-1}	$-\phi^{-1}$
ϕ^{-1}	-1	0	$-\phi^{-2}$	$-\phi$
1	$-\phi^{-1}$	ϕ^{-2}	0	$-2\phi^{-1}$
$\sqrt{5}$	ϕ^{-1}	ϕ	$2\phi^{-1}$	0

/	ϕ	ϕ^{-1}	$\sqrt{5}$
ϕ	1	ϕ^2	$(\sqrt{5}/5)\phi$
ϕ^{-1}	ϕ^{-2}	1	$(\sqrt{5}/5)\phi^{-1}$
$\sqrt{5}$	$\sqrt{5}\phi^{-1}$	$\sqrt{5}\phi$	1

Table of the first six exact powers of the number ϕ

n	ϕ^n	$(-\phi)^{-n}$	n	ϕ^n	$(-\phi)^{-n}$
0	1	1			
1	$(\sqrt{5}+1)/2$	$(-\sqrt{5}+1)/2$	-1	$(\sqrt{5}-1)/2$	$(-\sqrt{5}-1)/2$
2	$(\sqrt{5}+3)/2$	$(-\sqrt{5}+3)/2$	-2	$(-\sqrt{5}+3)/2$	$(\sqrt{5}+3)/2$
3	$\sqrt{5}+2$	$-\sqrt{5}+2$	-3	$\sqrt{5}-2$	$-\sqrt{5}-2$
4	$(3\sqrt{5}+7)/2$	$(-3\sqrt{5}+7)/2$	-4	$(-3\sqrt{5}+7)/2$	$(3\sqrt{5}+7)/2$
5	$(5\sqrt{5}+11)/2$	$(-5\sqrt{5}+11)/2$	-5	$(5\sqrt{5}-11)/2$	$(-5\sqrt{5}-11)/2$
6	$4\sqrt{5}+9$	$-4\sqrt{5}+9$	-6	$-4\sqrt{5}+9$	$4\sqrt{5}+9$

Table of the first six approximate powers of the number ϕ

n	approx(ϕ^n)	approx$(-\phi)^{-n}$	n	approx(ϕ^n)	approx$(-\phi)^{-n}$
0	1	1			
1	1.618033…	-0.618033…	-1	0.618033…	-1.618033…
2	2.618033…	0.381966…	-2	0.381966…	2.618033…
3	4.236067…	-0.236067…	-3	0.236067…	-4.236067…
4	6.854101…	0.145898…	-4	0.145898…	6.854101…
5	11.090169…	-0.090169…	-5	0.090169…	-11.090169…
6	17.944271…	0.055728…	-6	0.055728…	17.944271…

Table of the first six exact powers of the number ϕ multiplied with 2, $\sqrt{5}$ and $2\sqrt{5}$

n	$2\phi^n$	$\sqrt{5}\phi^n$	$2\sqrt{5}\phi^n$	n	$2\phi^n$	$\sqrt{5}\phi^n$	$2\sqrt{5}\phi^n$
0	2	$\sqrt{5}$	$2\sqrt{5}$				
1	$\sqrt{5}+1$	$(\sqrt{5}+5)/2$	$\sqrt{5}+5$	-1	$\sqrt{5}-1$	$(-\sqrt{5}+5)/2$	$-\sqrt{5}+5$
2	$\sqrt{5}+3$	$(3\sqrt{5}+5)/2$	$3\sqrt{5}+5$	-2	$-\sqrt{5}+3$	$(3\sqrt{5}-5)/2$	$3\sqrt{5}-5$
3	$2\sqrt{5}+4$	$2\sqrt{5}+5$	$4\sqrt{5}+10$	-3	$2\sqrt{5}-4$	$-2\sqrt{5}+5$	$-4\sqrt{5}+10$
4	$3\sqrt{5}+7$	$(7\sqrt{5}+15)/2$	$7\sqrt{5}+15$	-4	$-3\sqrt{5}+7$	$(7\sqrt{5}-15)/2$	$7\sqrt{5}-15$
5	$5\sqrt{5}+11$	$(11\sqrt{5}+25)/2$	$11\sqrt{5}+25$	-5	$5\sqrt{5}-11$	$(-11\sqrt{5}+25)/2$	$-11\sqrt{5}+25$
6	$8\sqrt{5}+18$	$9\sqrt{5}+20$	$18\sqrt{5}+40$	-6	$-8\sqrt{5}+18$	$9\sqrt{5}-20$	$18\sqrt{5}-40$

Definitions of the Fibonacci and the Lucas numbers

$$F(n) = (\phi^n-(-\phi)^{-n})/\sqrt{5} \qquad (2)$$

$n\in Z \Rightarrow F(n)\in Z$
$n\in\{R,C\} \Rightarrow F(n)\in C$

$$F_R(n) = (\phi^n-\cos(n\pi)\phi^{-n})/\sqrt{5} \qquad (2.1)^*$$

$n\in Z \Rightarrow F_R(n)\in Z$
$n\in R \Rightarrow F_R(n)\in R$

$$F(n+2) = F(n+1)+F(n),\ F(1)=1,\ F(0)=0 \qquad (3)$$

$n,k\in Z \Rightarrow F(n+k)\in Z$
$n,k\in\{R,C\} \Rightarrow F(n+k)\in C$

$$F(-n) = (-1)^{n+1}F(n) \qquad (3a)$$

$n\in Z \Rightarrow F(-n)\in Z$

[*] $F_R(n)$ = real part of $F(n)$, $n\in\{Z,R\}$; $n\in Z \Rightarrow F(n)=F_R(n)$, $n\in R \Rightarrow F(n)\neq F_R(n)$

$$L(n) = \phi^{n}+(-\phi)^{-n}$$ $n\in Z \Rightarrow L(n)\in Z$; $n\in\{R,C\} \Rightarrow L(n)\in C$ (4)

$$L_R(n) = \phi^{n}+\cos(n\pi)\phi^{-n}$$ $n\in Z \Rightarrow L_R(n)\in Z$; $n\in R \Rightarrow L_R(n)\in R$ (4.1)[*]

$$L(n+2) = L(n+1)+L(n),\ L(1)=1,\ L(0)=2$$ $n,k\in Z \Rightarrow L(n+k)\in Z$; $n,k\in\{R,C\} \Rightarrow L(n+k)\in C$ (5)

$$L(-n) = (-1)^{n}L(n)$$ $n\in Z \Rightarrow L(-n)\in Z$ (5a)

Another form of definitions of theFibonacci and the Lucas numbers is interesting because they contain four well known mathematical constants: e, ϕ, i and π:

$$F(n) = (\phi^{n}-e^{n\pi i}\phi^{-n})/\sqrt{5} = (\phi^{n}-(\cos(n\pi)+i*\sin(n\pi))\phi^{-n})/\sqrt{5}$$ $n\in Z \Rightarrow F(n)\in Z$; $n\in\{R,C\} \Rightarrow F(n)\in C$ (2a)

$$L(n) = \phi^{n}+e^{n\pi i}\phi^{-n} = \phi^{n}+(\cos(n\pi)+i*\sin(n\pi))\phi^{-n}$$ $n\in Z \Rightarrow L(n)\in Z$; $n\in\{R,C\} \Rightarrow L(n)\in C$ (4a)

Table of first sixteen Fibonacci and Lucas numbers

n	F(n)	L(n)	n	F(n)	L(n)
0	0	2			
1	1	1	-1	1	-1
2	1	3	-2	-1	3
3	2	4	-3	2	-4
4	3	7	-4	-3	7
5	5	11	-5	5	-11
6	8	18	-6	-8	18
7	13	29	-7	13	-29
8	21	47	-8	-21	47
9	34	76	-9	34	-76
10	55	123	-10	-55	123
11	89	199	-11	89	-199
12	144	322	-12	-144	322
13	233	521	-13	233	-521
14	377	843	-14	-377	843
15	610	1364	-15	610	-1364
16	987	2207	-16	-987	2207

Some connections between the Fibonacci and the Lucas numbers

$$L(n)^2-5F(n)^2 = 4(-1)^{n}$$ $n\in\{Z,R,C\}$ (6)

$$L_R(n)^2-5F_R(n)^2 = 4\cos(n\pi)$$ $n\in\{Z,R\}$ (6.1)

$$F(n) = \mathrm{sign}(n)^{n+1}*\sqrt{((L^2(n)-4(-1)^{n})/5)}$$ $n\in Z$ (7)

[*] $L_R(n)$ = real part of $L(n)$, $n\in\{Z,R\}$; $n\in Z \Rightarrow L(n)=L_R(n)$, $n\in R \Rightarrow L(n)\neq L_R(n)$

$L(n) = \mathrm{sign}(n)^n*\sqrt{(5F^2(n)+4(-1)^n)}$	$n\in Z$	(8)
$F(2n) = F(n)L(n)$	$n\in\{Z,R,C\}$	(9)
$L(2n) = L(n)^2-2(-1)^n$	—||—	(10)

Formulas with the Golden Mean

$\phi^n = \phi^{n-1}+\phi^{n-2}$	$n\in\{Z,R,C\}$	(11)
$\phi^n = \phi^{n+1}-\phi^{n-1}$	—||—	(11a)
$\phi^n = \phi^{n+2}-\phi^{n+1}$	—||—	(11b)
$L(2k-1)\phi^{n+2k-1} = \phi^{n+4k-2}-\phi^n$	$n\in\{Z,R,C\}$, $k\in Z$	(12)
$\phi^{n+1} = \phi^{n+2}-\phi^n$	$k=1$	(12a)
$4\phi^{n+3} = \phi^{n+6}-\phi^n$	$k=2$	(12b)
$F(2k)\sqrt{5}\phi^{n+2k} = \phi^{n+4k}-\phi^n$	$n\in\{Z,R,C\}$, $k\in Z$	(13)
$\sqrt{5}\phi^{n+2} = \phi^{n+4}-\phi^n$	$k=1$	(13a)
$3\sqrt{5}\phi^{n+4} = \phi^{n+8}-\phi^n$	$k=2$	(13b)
$F(2k-1)\sqrt{5}\phi^{n+2k-1} = \phi^{n+4k-2}+\phi^n$	$n\in\{Z,R,C\}$, $k\in Z$	(14)
$\sqrt{5}\phi^{n+1} = \phi^{n+2}+\phi^n$	$k=1$	(14a)
$2\sqrt{5}\phi^{n+3} = \phi^{n+6}+\phi^n$	$k=2$	(14b)
$L(2k)\phi^{n+2k} = \phi^{n+4k}+\phi^n$	$n\in\{Z,R,C\}$, $k\in Z$	(15)
$3\phi^{n+2} = \phi^{n+4}+\phi^n$	$k=1$	(15a)
$7\phi^{n+4} = \phi^{n+8}+\phi^n$	$k=2$	(15b)

Formulas with the Golden Mean and the Fibonacci numbers

$F(k)\phi^n = F(n)\phi^k-(-1)^kF(n-k)$	$k,n\in\{Z,R,C\}$	(16)
$\phi^n = F(n)\phi^2-F(n-2)$	$k=2$	(16a)
$\phi^n = F(n)\phi+F(n-1)$	$k=1$	(16b)
$\phi^n = F(n)\phi^{-1}+F(n+1)$	$k=-1$	(16d)
$\phi^n = -F(n)\phi^{-2}+F(n+2)$	$k=-2$	(16e)
$F(n-1)\phi^n = F(n)\phi^{n-1}+(-1)^n$	$k=n-1$	(16f)
$F(n)\phi^{2n} = F(2n)\phi^n-(-1)^nF(n)$	$n=2k$	(16g)
$F_R(k)\phi^n = F_R(n)\phi^k-\cos(k\pi)F_R(n-k)$	$k,n\in\{Z,R\}$, $n-k\in Z$	(16.1)
$F_R(k+1/2)\phi^{n+1/2} = F_R(n+1/2)\phi^{k+1/2}$	$k,n\in Z$	(16.1a)
$F(k)\phi^n = F(n)\phi^k+(-1)^nF(k-n)$	$k,n\in\{Z,R,C\}$	(17)
$F_R(k)\phi^n = F_R(n)\phi^k+\cos(n\pi)F_R(k-n)$	$k,n\in\{Z,R\}$, $k-n\in Z$	(17.1)
$F(k)(-\phi)^{-n} = -F(n)\phi^k+F(n+k)$	$k,n\in\{Z,R,C\}$	(18)
$(-\phi)^{-n} = -F(n)\phi+F(n+1)$	$k=1$	(18b)
$(-\phi)^{-n} = -F(n)\phi^{-1}+F(n-1)$	$k=-1$	(18d)
$\phi^n = (-\phi)^{-n}+F(n)\sqrt{5}$	$n\in\{Z,R,C\}$	(19)
$(-\phi)^{-n} = \phi^n-F(n)\sqrt{5}$	—‖—	(19a)
$\phi^n = \cos(n\pi)\phi^{-n}+F_R(n)\sqrt{5}$	$n\in\{Z,R\}$	(19.1)
$2\sqrt{5}\phi^n = 5(\mathrm{sign}(n)^n*\sqrt{(F^2(n)+4/5(-1)^n)}+F(n))$	$n\in Z$	(20)
$\phi^{2n} = \sqrt{(5F^2(n)+2(1+(-1)^n))}\phi^n -1$	—‖—	(21)
$\phi^{2n} = \mathrm{sign}(n)\sqrt{(5F^2(n)-2(1-(-1)^n))}\phi^n +1$	—‖—	(22)

Formulas with the Golden Mean and the Lucas numbers

$F(k)\sqrt{5}\phi^{n} = L(n)\phi^{k}-(-1)^{k}L(n-k)$	$k,n\in\{Z,R,C\}$	(23)
$\sqrt{5}\phi^{n} = L(n)\phi^{2}-L(n-2)$	$k=2$	(23a)
$\sqrt{5}\phi^{n} = L(n)\phi+L(n-1)$	$k=1$	(23b)
$\sqrt{5}\phi^{n} = L(n)\phi^{-1}+L(n+1)$	$k=-1$	(23d)
$\sqrt{5}\phi^{n} = -L(n)\phi^{-2}+L(n+2)$	$k=-2$	(23e)
$F(n-1)\sqrt{5}\phi^{n} = L(n)\phi^{n-1}+(-1)^{n}$	$k=n-1$	(23f)
$F(n)\sqrt{5}\phi^{2n} = L(2n)\phi^{n}-(-1)^{n}L(n)$	$n=2k$	(23g)
$F_{R}(k)\sqrt{5}\phi^{n} = L_{R}(n)\phi^{k}-\cos(k\pi)L_{R}(n-k)$	$k,n\in\{Z,R\}$, $n-k\in Z$	(23.1)
$F_{R}(k+1/2)\sqrt{5}\phi^{n+1/2} = L_{R}(n+1/2)\phi^{k+1/2}$	$k,n\in Z$	(23.1a)
$F(k)\sqrt{5}\phi^{n} = L(n)\phi^{k}-(-1)^{n}L(k-n)$	$k,n\in\{Z,R,C\}$	(24)
$F_{R}(k)\sqrt{5}\phi^{n} = L_{R}(n)\phi^{k}-\cos(n\pi)L_{R}(k-n)$	$k,n\in\{Z,R\}$, $k-n\in Z$	(24.1)
$F(k)\sqrt{5}(-\phi)^{-n} = L(n)\phi^{k}-L(n+k)$	$k,n\in\{Z,R,C\}$	(25)
$\sqrt{5}(-\phi)^{-n} = L(n)\phi-L(n+1)$	$k=1$	(25b)
$\sqrt{5}(-\phi)^{-n} = L(n)\phi^{-1}-L(n-1)$	$k=-1$	(25d)
$\phi^{n} = -(-\phi)^{-n}+L(n)$	$n\in\{Z,R,C\}$	(26)
$(-\phi)^{-n} = -\phi^{n}+L(n)$	—‖—	(26a)
$\phi^{n} = -\cos(n\pi)\phi^{-n}+L_{R}(n)$	$n\in\{Z,R\}$	(26.1)
$2\phi^{n} = \mathrm{sign}(n)^{n+1}*\sqrt{(L^{2}(n)-4(-1)^{n})} +L(n)$	$n\in Z$	(27)
$\phi^{2n} = \sqrt{(L^{2}(n)+2(1-(-1)^{n}))}\phi^{n} -1$	—‖—	(28)
$\phi^{2n} = \mathrm{sign}(n)\sqrt{(L^{2}(n)-2(1+(-1)^{n}))}\phi^{n} +1$	—‖—	(29)

Formulas with the Golden Mean, the Fibonacci and the Lucas numbers

$L(k)\phi^n = F(n)\sqrt{5}\phi^k+(-1)^kL(n-k)$	$k,n\in\{Z,R,C\}$	(30)
$3\phi^n = F(n)\sqrt{5}\phi^2+L(n-2)$	$k=2$	(30a)
$\phi^n = F(n)\sqrt{5}\phi-L(n-1)$	$k=1$	(30b)
$2\phi^n = F(n)\sqrt{5} +L(n)$	$k=0$	(30c)
$\phi^n = -F(n)\sqrt{5}\phi^{-1}+L(n+1)$	$k=-1$	(30d)
$3\phi^n = F(n)\sqrt{5}\phi^{-2}+L(n+2)$	$k=-2$	(30e)
$L(n-1)\phi^n = F(n)\sqrt{5}\phi^{n-1}-(-1)^n$	$k=n-1$	(30f)
$L(n)\phi^{2n} = F(2n)\sqrt{5}\phi^n+(-1)^nL(n)$	$n=2k$	(30g)
$L_R(k)\phi^n = F_R(n)\sqrt{5}\phi^k+\cos(k\pi)L_R(n-k)$	$k,n\in\{Z,R\}$, $n-k\in Z$	(30.1)
$L_R(k+1/2)\phi^{n+1/2} = F_R(n+1/2)\sqrt{5}\phi^{k+1/2}$	$k,n\in Z$	(30.1a) = (23.1a)
$L(k)\phi^n = F(n)\sqrt{5}\phi^k+(-1)^nL(k-n)$	$k,n\in\{Z,R,C\}$	(31)
$L_R(k)\phi^n = F_R(n)\sqrt{5}\phi^k+\cos(n\pi)L_R(k-n)$	$k,n\in\{Z,R\}$, $k-n\in Z$	(31.1)
$L(k)(-\phi)^{-n} = -F(n)\sqrt{5}\phi^k+L(n+k)$	$k,n\in\{Z,R,C\}$	(32)
$(-\phi)^{-n} = -F(n)\sqrt{5}\phi+L(n+1)$	$k=1$	(32b)
$2(-\phi)^{-n} = -F(n)\sqrt{5} +L(n)$	$k=0$	(32c)
$(-\phi)^{-n} = F(n)\sqrt{5}\phi^{-1}-L(n-1)$	$k=-1$	(32d)
$L(k)\phi^n = L(n)\phi^k+(-1)^kF(n-k)\sqrt{5}$	$k,n\in\{Z,R,C\}$	(33)
$3\phi^n = L(n)\phi^2+F(n-2)\sqrt{5}$	$k=2$	(33a)
$\phi^n = L(n)\phi- F(n-1)\sqrt{5}$	$k=1$	(33b)
$2\phi^n = L(n) +F(n)\sqrt{5}$	$k=0$	(33c)=(30c)
$\phi^n = -L(n)\phi^{-1}+F(n+1)\sqrt{5}$	$k=-1$	(33d)

$3\phi^{n} = L(n)\phi^{-2}+F(n+2)\sqrt{5}$	k=-2	(33e)
$L(n-1)\phi^{n} = L(n)\phi^{n-1}-(-1)^{n}\sqrt{5}$	k=n-1	(33f)
$L(n)\phi^{2n} = L(2n)\phi^{n}+(-1)^{n}F(n)\sqrt{5}$	n=2k	(33g)
$L_R(k)\phi^{n} = L_R(n)\phi^{k}+\cos(k\pi)F_R(n-k)\sqrt{5}$	$k,n\in\{Z,R\}$, $n-k\in Z$	(33.1)
$L_R(k+1/2)\phi^{n+1/2} = L_R(n+1/2)\phi^{k+1/2}$	$k,n\in Z$	(33.1a)
$L(k)\phi^{n} = L(n)\phi^{k}-(-1)^{n}F(k-n)\sqrt{5}$	$k,n\in\{Z,R,C\}$	(34)
$L_R(k)\phi^{n} = L_R(n)\phi^{k}- \cos(n\pi)F_R(k-n)\sqrt{5}$	$k,n\in\{Z,R\}$, $k-n\in Z$	(34.1)
$L(k)(-\phi)^{-n} = L(n)\phi^{k}-F(n+k)\sqrt{5}$	$k,n\in\{Z,R,C\}$	(35)
$(-\phi)^{-n} = L(n)\phi-F(n+1)\sqrt{5}$	k=1	(35b)
$2(-\phi)^{-n} = L(n)-F(n)\sqrt{5}$	k=0	(35c)=(32c)
$(-\phi)^{-n} = -L(n)\phi^{-1}+F(n-1)\sqrt{5}$	k=-1	(35d)
$2\sqrt{5}\phi^{n} = L(n)\sqrt{5}+5F(n)$	$n\in\{Z,R,C\}$	(36)

Analogies

(2)≡(2a)	(4)≡(4a)	$n\in\{Z,R,C\}$
(2)≡(2a)≡(2.1)	(4)≡(4a)≡(4.1)	$n\in Z$
(16)≡(17)	(23)≡(24)	$k,n\in\{Z,R,C\}$
(16)≡(16.1)≡(17)≡(17.1)	(23)≡(23.1)≡(24)≡(24.1)	$k,n\in Z$
(30)≡(31)	(33)≡(34)	$k,n\in\{Z,R,C\}$
(30)≡(30.1)≡(31)≡(31.1)	(33)≡(33.1)≡(34)≡(34.1)	$k,n\in Z$
(19)≡(19.1)	(26)≡(26.1)	$n\in Z$

Formulas with the Golden Mean and trigonometric functions

$\phi = 1/(2\sin(\pi/10))$ (37)

$\phi = 2\sin(3\pi/10)$ (38)

$\phi = 2\cos(\pi/5)$ (39)

$\phi = 1/(2\cos(2\pi/5)$ (40)

Formulas with the Golden Mean and inverse trigonometric functions

$\phi^n = e^{-i*n*\mathrm{arcsin}(i/2)}$ $n\in\{Z,R,C\}$ (41)

$\phi^n = (-i)^n e^{i*n*\mathrm{arccos}(i/2)}$ —||— (42)

$\phi^n = e^{i*n*\mathrm{arctan}(-i\sqrt{(5)}/5)}$ —||— (43)

$\phi^n = (i)^n e^{-i*n*\mathrm{arcsec}(2i)}$ —||— (44)

$\phi^n = e^{i*n*\mathrm{arccsc}(2i)}$ —||— (45)

$\phi^n = (i)^n e^{-i*n*\mathrm{arccot}(-i\sqrt{(5)}/5)}$ —||— (46)

Formulas with the Golden Mean and inverse hyperbolic functions

$\phi^n = e^{n*\mathrm{arcsinh}(1/2)}$ $n\in\{Z,R,C\}$ (47)

$\phi^n = e^{n*\mathrm{arccosh}(\sqrt{(5)}/2)}$ —||— (48)

$\phi^n = e^{n*\mathrm{arctanh}(\sqrt{(5)}/5)}$ —||— (49)

$\phi^n = e^{n*\mathrm{arcsech}(2/\sqrt{5})}$ —||— (50)

$\phi^n = e^{n*\mathrm{arccsch}(2)}$ —||— (51)

$\phi^n = e^{n*\mathrm{arccoth}(\sqrt{5})}$ —||— (52)

Conclusion

The formulas (16), (23), (30) and (33) are very interesting because they present the connections between any two powers of the number ϕ.

Any power of the number ϕ can be expressed in an infinite number of formulas with the other powers of the number ϕ in the form $m*\phi^n = p*\phi^k+r$ where the variables m, p and r are the Fibonacci or the Lucas numbers or the Fibonacci numbers multiplied with $\sqrt{5}$. Variables n and k can be integers, real numbers or complex numbers.

Examples for formulas (16), (23), (30) and (33) for n=2 and k={3, 2, 1, 0, -1, -2, -3}

k	(16)	(23)	(30)	(33)
3	$2\phi^2 = \phi^3+1$	$2\sqrt{5}\phi^2 = 3\phi^3-1$	$4\phi^2 = \sqrt{5}\phi^3+1$	$4\phi^2 = 3\phi^3-\sqrt{5}$
2	$\phi^2 = \phi^2$	$\sqrt{5}\phi^2 = 3\phi^2-2$	$3\phi^2 = \sqrt{5}\phi^2+2$	$3\phi^2 = 3\phi^2$
1	$\phi^2 = \phi+1$	$\sqrt{5}\phi^2 = 3\phi+1$	$\phi^2 = \sqrt{5}\phi-1$	$\phi^2 = 3\phi-\sqrt{5}$
0			$2\phi^2 = \sqrt{5}+3$	$2\phi^2 = 3+\sqrt{5}$
-1	$\phi^2 = \phi^{-1}+2$	$\sqrt{5}\phi^2 = 3\phi^{-1}+4$	$-\phi^2 = \sqrt{5}\phi^{-1}-4$	$-\phi^2 = 3\phi^{-1}-2\sqrt{5}$
-2	$-\phi^2 = \phi^{-2}-3$	$-\sqrt{5}\phi^2 = 3\phi^{-2}-7$	$3\phi^2 = \sqrt{5}\phi^{-2}+7$	$3\phi^2 = 3\phi^{-2}+3\sqrt{5}$
-3	$2\phi^2 = \phi^{-3}+5$	$2\sqrt{5}\phi^2 = 3\phi^{-3}+11$	$-4\phi^2 = \sqrt{5}\phi^{-3}-11$	$-4\phi^2 = 3\phi^{-3}-5\sqrt{5}$

Abbreviations:

ϕ	Golden Mean number Phi, see (1)
F(n)	n_th Fibonacci number, see (2)
F_R(n)	real part of the n_th Fibonacci number, see (2.1)
L(n)	n_th Lucas number, see (4)
L_R(n)	real part of the n_th Lucas number, see (4.1)
C	complex number
R	real number
Z	integer

Appendix

Selected examples for formulas (11)...(36), $k,n \in Z$

	(11)	(11a), (12a)	(11b)
$\phi^3 =$	$\phi^2+\phi$	$\phi^4-\phi^2$	$\phi^5-\phi^4$
$\phi^2 =$	$\phi+1$	$\phi^3-\phi$	$\phi^4-\phi^3$
$\phi =$	$1+\phi^{-1}$	ϕ^2-1	$\phi^3-\phi^2$
$1 =$	$\phi^{-1}+\phi^{-2}$	$\phi-\phi^{-1}$	$\phi^2-\phi$
$\phi^{-1} =$	$\phi^{-2}+\phi^{-3}$	$1-\phi^{-2}$	$\phi-1$
$\phi^{-2} =$	$\phi^{-3}+\phi^{-4}$	$\phi^{-1}-\phi^{-3}$	$1-\phi^{-1}$
$\phi^{-3} =$	$\phi^{-4}+\phi^{-5}$	$\phi^{-2}-\phi^{-4}$	$\phi^{-1}-\phi^{-2}$

	(16a)	(16b)	(16d)	(16e)	(19)	(26)
$\phi^3 =$	$2\phi^2-1$	$2\phi+1$	$2\phi^{-1}+3$	$-2\phi^{-2}+5$	$-\phi^{-3}+2\sqrt{5}$	$\phi^{-3}+4$
$\phi^2 =$	ϕ^2	$\phi+1$	$\phi^{-1}+2$	$-\phi^{-2}+3$	$\phi^{-2}+\sqrt{5}$	$-\phi^{-2}+3$
$\phi =$	ϕ^2-1	ϕ	$\phi^{-1}+1$	$-\phi^{-2}+2$	$-\phi^{-1}+\sqrt{5}$	$\phi^{-1}+1$
$1 =$	1	1	1	1	1	1
$\phi^{-1} =$	ϕ^2-2	$\phi-1$	ϕ^{-1}	$-\phi^{-2}+1$	$-\phi+\sqrt{5}$	$\phi-1$
$\phi^{-2} =$	$-\phi^2+3$	$-\phi+2$	$-\phi^{-1}+1$	ϕ^{-2}	$\phi^2-\sqrt{5}$	$-\phi^2+3$
$\phi^{-3} =$	$2\phi^2-5$	$2\phi-3$	$2\phi^{-1}-1$	$-2\phi^{-2}+1$	$-\phi^3+2\sqrt{5}$	ϕ^3-4

	(30b)	(30d)	(33b)	(33d)
$\phi^3 =$	$2\sqrt{5}\phi-3$	$-2\sqrt{5}\phi^{-1}+7$	$4\phi-\sqrt{5}$	$-4\phi^{-1}+3\sqrt{5}$
$\phi^2 =$	$\sqrt{5}\phi-1$	$-\sqrt{5}\phi^{-1}+4$	$3\phi-\sqrt{5}$	$-3\phi^{-1}+2\sqrt{5}$
$\phi =$	$\sqrt{5}\phi-2$	$-\sqrt{5}\phi^{-1}+3$	ϕ	$-\phi^{-1}+\sqrt{5}$
$1 =$	1	1	$2\phi-\sqrt{5}$	$-2\phi^{-1}+\sqrt{5}$
$\phi^{-1} =$	$\sqrt{5}\phi-3$	$-\sqrt{5}\phi^{-1}+2$	$-\phi+\sqrt{5}$	ϕ^{-1}
$\phi^{-2} =$	$-\sqrt{5}\phi+4$	$\sqrt{5}\phi^{-1}-1$	$3\phi-2\sqrt{5}$	$-3\phi^{-1}+\sqrt{5}$
$\phi^{-3} =$	$2\sqrt{5}\phi-7$	$-2\sqrt{5}\phi^{-1}+3$	$-4\phi+3\sqrt{5}$	$4\phi^{-1}-\sqrt{5}$

	(15a)	(30a)	(30e)	(33a)	(33e)
$3\phi^3 =$	$\phi^5+\phi$	$2\sqrt{5}\phi^2+1$	$2\sqrt{5}\phi^{-2}+11$	$4\phi^2+\sqrt{5}$	$4\phi^{-2}+5\sqrt{5}$
$3\phi^2 =$	ϕ^4+1	$\sqrt{5}\phi^2+2$	$\sqrt{5}\phi^{-2}+7$	$3\phi^2$	$3\phi^{-2}+3\sqrt{5}$
$3\phi =$	$\phi^3+\phi^{-1}$	$\sqrt{5}\phi^2-1$	$\sqrt{5}\phi^{-2}+4$	$\phi^2+\sqrt{5}$	$\phi^{-2}+2\sqrt{5}$
$3 =$	$\phi^2+\phi^{-2}$	3	3	$2\phi^2-\sqrt{5}$	$2\phi^{-2}+\sqrt{5}$
$3\phi^{-1} =$	$\phi+\phi^{-3}$	$\sqrt{5}\phi^2-4$	$\sqrt{5}\phi^{-2}+1$	$-\phi^2+2\sqrt{5}$	$-\phi^{-2}+\sqrt{5}$
$3\phi^{-2} =$	$1+\phi^{-4}$	$-\sqrt{5}\phi^2+7$	$-\sqrt{5}\phi^{-2}+2$	$3\phi^2-3\sqrt{5}$	ϕ^{-2}
$3\phi^{-3} =$	$\phi^{-1}+\phi^{-5}$	$2\sqrt{5}\phi^2-11$	$2\sqrt{5}\phi^{-2}-1$	$-4\phi^2+5\sqrt{5}$	$-4\phi^{-2}+\sqrt{5}$

	(27), (30c)
$2\phi^3 =$	$2\sqrt{5}+4$
$2\phi^2 =$	$\sqrt{5}+3$
$2\phi =$	$\sqrt{5}+1$
$2 =$	2
$2\phi^{-1} =$	$\sqrt{5}-1$
$2\phi^{-2} =$	$-\sqrt{5}+3$
$2\phi^{-3} =$	$2\sqrt{5}-4$

	(12b)
$4\phi^3 =$	ϕ^6-1
$4\phi^2 =$	$\phi^5-\phi^{-1}$
$4\phi =$	$\phi^4-\phi^{-2}$
$4 =$	$\phi^3-\phi^{-3}$
$4\phi^{-1} =$	$\phi^2-\phi^{-4}$
$4\phi^{-2} =$	$\phi-\phi^{-5}$
$4\phi^{-3} =$	$1-\phi^{-6}$

	(15b)
$7\phi^3 =$	$\phi^7+\phi^{-1}$
$7\phi^2 =$	$\phi^6+\phi^{-2}$
$7\phi =$	$\phi^5+\phi^{-3}$
$7 =$	$\phi^4+\phi^{-4}$
$7\phi^{-1} =$	$\phi^3+\phi^{-5}$
$7\phi^{-2} =$	$\phi^2+\phi^{-6}$
$7\phi^{-3} =$	$\phi+\phi^{-7}$

	(13a)	(14a)	(23a)	(23b)	(23d)	(23e)
$\sqrt{5}\phi^3 =$	$\phi^5-\phi$	$\phi^4+\phi^2$	$4\phi^2-1$	$4\phi+3$	$4\phi^{-1}+7$	$-4\phi^{-2}+11$
$\sqrt{5}\phi^2 =$	ϕ^4-1	$\phi^3+\phi$	$3\phi^2-2$	$3\phi+1$	$3\phi^{-1}+4$	$-3\phi^{-2}+7$
$\sqrt{5}\phi =$	$\phi^3-\phi^{-1}$	ϕ^2+1	ϕ^2+1	$\phi+2$	$\phi^{-1}+3$	$-\phi^{-2}+4$
$\sqrt{5} =$	$\phi^2-\phi^{-2}$	$\phi+\phi^{-1}$	$2\phi^2-3$	$2\phi-1$	$2\phi^{-1}+1$	$-2\phi^{-2}+3$
$\sqrt{5}\phi^{-1} =$	$\phi-\phi^{-3}$	$1+\phi^{-2}$	$-\phi^2+4$	$-\phi+3$	$-\phi^{-1}+2$	$\phi^{-2}+1$
$\sqrt{5}\phi^{-2} =$	$1-\phi^{-4}$	$\phi^{-1}+\phi^{-3}$	$3\phi^2-7$	$3\phi-4$	$3\phi^{-1}-1$	$-3\phi^{-2}+2$
$\sqrt{5}\phi^{-3} =$	$\phi^{-1}-\phi^{-5}$	$\phi^{-2}+\phi^{-4}$	$-4\phi^2+11$	$-4\phi+7$	$-4\phi^{-1}+3$	$4\phi^{-2}-1$

	(14b)	(20), (36)
$2\sqrt{5}\phi^3 =$	ϕ^6+1	$4\sqrt{5}+10$
$2\sqrt{5}\phi^2 =$	$\phi^5+\phi^{-1}$	$3\sqrt{5}+5$
$2\sqrt{5}\phi =$	$\phi^4+\phi^{-2}$	$\sqrt{5}+5$
$2\sqrt{5} =$	$\phi^3+\phi^{-3}$	$2\sqrt{5}$
$2\sqrt{5}\phi^{-1} =$	$\phi^2+\phi^{-4}$	$-\sqrt{5}+5$
$2\sqrt{5}\phi^{-2} =$	$\phi+\phi^{-5}$	$3\sqrt{5}-5$
$2\sqrt{5}\phi^{-3} =$	$1+\phi^{-6}$	$-4\sqrt{5}+10$

	(13b)
$3\sqrt{5}\phi^3 =$	$\phi^7-\phi^{-1}$
$3\sqrt{5}\phi^2 =$	$\phi^6-\phi^{-2}$
$3\sqrt{5}\phi =$	$\phi^5-\phi^{-3}$
$3\sqrt{5} =$	$\phi^4-\phi^{-4}$
$3\sqrt{5}\phi^{-1} =$	$\phi^3-\phi^{-5}$
$3\sqrt{5}\phi^{-2} =$	$\phi^2-\phi^{-6}$
$3\sqrt{5}\phi^{-3} =$	$\phi-\phi^{-7}$

	(18b)	(18d)	(19a)	(26a)
$-\phi^{-3} =$	$-2\phi+3$	$-2\phi^{-1}+1$	$\phi^3-2\sqrt{5}$	$-\phi^3+4$
$\phi^{-2} =$	$-\phi+2$	$-\phi^{-1}+1$	$\phi^2-\sqrt{5}$	$-\phi^2+3$
$-\phi^{-1} =$	$-\phi+1$	$-\phi^{-1}$	$\phi-\sqrt{5}$	$-\phi+1$
$1 =$	1	1	1	1
$-\phi =$	$-\phi$	$-\phi^{-1}-1$	$\phi^{-1}-\sqrt{5}$	$-\phi^{-1}-1$
$\phi^2 =$	$\phi+1$	$\phi^{-1}+2$	$\phi^{-2}+\sqrt{5}$	$-\phi^{-2}+3$
$-\phi^3 =$	$-2\phi-1$	$-2\phi^{-1}-3$	$\phi^{-3}-2\sqrt{5}$	$-\phi^{-3}-4$

	(32b)	(32d)	(35b)	(35d)
$-\phi^{-3} =$	$-2\sqrt{5}\phi+7$	$2\sqrt{5}\phi^{-1}-3$	$4\phi-3\sqrt{5}$	$-4\phi^{-1}+\sqrt{5}$
$\phi^{-2} =$	$-\sqrt{5}\phi+4$	$\sqrt{5}\phi^{-1}-1$	$3\phi-2\sqrt{5}$	$-3\phi^{-1}+\sqrt{5}$
$-\phi^{-1} =$	$-\sqrt{5}\phi+3$	$\sqrt{5}\phi^{-1}-2$	$\phi-\sqrt{5}$	$-\phi^{-1}$
$1 =$	1	1	$2\phi-\sqrt{5}$	$-2\phi^{-1}+\sqrt{5}$
$-\phi =$	$-\sqrt{5}\phi+2$	$\sqrt{5}\phi^{-1}-3$	$-\phi$	$\phi^{-1}-\sqrt{5}$
$\phi^2 =$	$\sqrt{5}\phi-1$	$-\sqrt{5}\phi^{-1}+4$	$3\phi-\sqrt{5}$	$-3\phi^{-1}+2\sqrt{5}$
$-\phi^3 =$	$-2\sqrt{5}\phi+3$	$2\sqrt{5}\phi^{-1}-7$	$-4\phi+\sqrt{5}$	$4\phi^{-1}-3\sqrt{5}$

	(32c)
$-2\phi^{-3} =$	$-2\sqrt{5}+4$
$2\phi^{-2} =$	$-\sqrt{5}+3$
$-2\phi^{-1} =$	$-\sqrt{5}+1$
$2 =$	2
$-2\phi =$	$-\sqrt{5}-1$
$2\phi^2 =$	$\sqrt{5}+3$
$-2\phi^3 =$	$-2\sqrt{5}-4$

	(25b)	(25d)
$-\sqrt{5}\phi^{-3} =$	$4\phi-7$	$4\phi^{-1}-3$
$\sqrt{5}\phi^{-2} =$	$3\phi-4$	$3\phi^{-1}-1$
$-\sqrt{5}\phi^{-1} =$	$\phi-3$	$\phi^{-1}-2$
$\sqrt{5} =$	$2\phi-1$	$2\phi^{-1}+1$
$-\sqrt{5}\phi =$	$-\phi-2$	$-\phi^{-1}-3$
$\sqrt{5}\phi^2 =$	$3\phi+1$	$3\phi^{-1}+4$
$-\sqrt{5}\phi^3 =$	$-4\phi-3$	$-4\phi^{-1}-7$

	(16f)
$\phi^3 =$	$2\phi^2-1$
$\phi^2 =$	$\phi+1$
$0 =$	0
$1 =$	1
$-\phi^{-1} =$	$\phi^{-2}-1$
$2\phi^{-2} =$	$-\phi^{-3}+1$
$-3\phi^{-3} =$	$2\phi^{-4}-1$

	(23f)
$\sqrt{5}\phi^3 =$	$4\phi^2-1$
$\sqrt{5}\phi^2 =$	$3\phi+1$
$0 =$	0
$\sqrt{5} =$	$2\phi^{-1}+1$
$-\sqrt{5}\phi^{-1} =$	$-\phi^{-2}-1$
$2\sqrt{5}\phi^{-2} =$	$3\phi^{-3}+1$
$-3\sqrt{5}\phi^{-3} =$	$-4\phi^{-4}-1$

	(30f)	(33f)
$3\phi^3 =$	$2\sqrt{5}\phi^2+1$	$4\phi^2+\sqrt{5}$
$\phi^2 =$	$\sqrt{5}\phi-1$	$3\phi-\sqrt{5}$
$2\phi =$	$\sqrt{5}+1$	$1+\sqrt{5}$
$-1 =$	-1	$2\phi^{-1}-\sqrt{5}$
$3\phi^{-1} =$	$\sqrt{5}\phi^{-2}+1$	$-\phi^{-2}+\sqrt{5}$
$-4\phi^{-2} =$	$-\sqrt{5}\phi^{-3}-1$	$3\phi^{-3}-\sqrt{5}$
$7\phi^{-3} =$	$2\sqrt{5}\phi^{-4}+1$	$-4\phi^{-4}+\sqrt{5}$

	(21), (28)	(22), (29)
$\phi^6 =$	$2\sqrt{5}\phi^3-1$	$4\phi^3+1$
$\phi^4 =$	$3\phi^2-1$	$\sqrt{5}\phi^2+1$
$\phi^2 =$	$\sqrt{5}\phi-1$	$\phi+1$
$1 =$	1	1
$\phi^{-2} =$	$\sqrt{5}\phi^{-1}-1$	$-\phi^{-1}+1$
$\phi^{-4} =$	$3\phi^{-2}-1$	$-\sqrt{5}\phi^{-2}+1$
$\phi^{-6} =$	$2\sqrt{5}\phi^{-3}-1$	$-4\phi^{-3}+1$

	(16g)
$2\phi^6 =$	$8\phi^3+2$
$\phi^4 =$	$3\phi^2-1$
$\phi^2 =$	$\phi+1$
$0 =$	0
$\phi^{-2} =$	$-\phi^{-1}+1$
$-\phi^{-4} =$	$-3\phi^{-2}+1$
$2\phi^{-6} =$	$-8\phi^{-3}+2$

	(23g)
$2\sqrt{5}\phi^6 =$	$18\phi^3+4$
$\sqrt{5}\phi^4 =$	$7\phi^2-3$
$\sqrt{5}\phi^2 =$	$3\phi+1$
$0 =$	0
$\sqrt{5}\phi^{-2} =$	$3\phi^{-1}-1$
$-\sqrt{5}\phi^{-4} =$	$7\phi^{-2}-3$
$2\sqrt{5}\phi^{-6} =$	$18\phi^{-3}-4$

	(30g)	(33g)
$4\phi^6 =$	$8\sqrt{5}\phi^3-4$	$18\phi^3-2\sqrt{5}$
$3\phi^4 =$	$3\sqrt{5}\phi^2+3$	$7\phi^2+\sqrt{5}$
$\phi^2 =$	$\sqrt{5}\phi-1$	$3\phi-\sqrt{5}$
$2 =$	2	2
$-\phi^{-2} =$	$-\sqrt{5}\phi^{-1}+1$	$3\phi^{-1}-\sqrt{5}$
$3\phi^{-4} =$	$-3\sqrt{5}\phi^{-2}+3$	$7\phi^{-2}-\sqrt{5}$
$-4\phi^{-6} =$	$-8\sqrt{5}\phi^{-3}+4$	$18\phi^{-3}-2\sqrt{5}$

Examples for analogous formulas (chapter Analogies) are identical and in the Appendix is stated usually only one formula. For example: formulas (16.1), (17) and (17.1) are not stated but they are analogous with formula (16) which has several examples (16a)…(16g).

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