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This program models incompressible irrotational two-dimensional flow around circles and airfoils.

We only consider steady flow, which means by definition that the vector field V of the flow is independent of time: $V = (u(x,y), v(x,y))$. Because the flow is irrotational, there exists a function Φ(x,y), the velocity potential, such that $\nabla\Phi = V$. The continuity equation

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ assures the existence of a stream function Ψ such that $u = \frac{\partial \Psi}{\partial y}, v = -\frac{\partial \Psi}{\partial x}$.

The lines of constant Ψ are streamlines, that is, lines parallel to the velocity V.

If we now set $W(z) = \Phi(x,y) + i\Psi(x,y)$, where $z = x + iy$, then incompressibility and irrotationality are equivalent to the Cauchy-Riemann equations for W. So W is a complex analytic function of z. We then get $W' = u - iv$. W is called the complex potential of V. Inversely, we can start with a complex analytic function W(z) and get an incompressible irrotational flow by setting $W' = u - iv, V = (u, v)$.

Let $c \in C$ $with$ $Re(c) \leq 0$ and $Im(c) \geq 0$ and $V_\infty, \Gamma \in \mathbb{R}, V_\infty > 0$. Let a be the radius of the circle with center c which passes through the point (1,0). Consider

$$W(z) \coloneqq V_\infty\left[(z-c)e^{-i\alpha} + \frac{a^2 e^{i\alpha}}{(z-c)^2}\right] + \frac{i\Gamma}{\pi}\ln\left(\frac{(z-c)e^{-i\alpha}}{a}\right).$$

Then W is the complex potential of the flow about a circle of radius a in the z-plane, the circle being centered at $z=c$, the angle of attack at infinity being α, the speed at infinity being V_∞, the circulation being Γ, which will be determined by the Kutta condition. We set $W' = u - iv, V = (u, v)$ and construct a number of flow lines (streamlines) $\varphi_k(t)$ for this vector field by choosing starting points $\varphi_k(0)$ outside the circle with different distances from the center c and solving the ordinary differential equation $\dot{\varphi}_k(t) = V(\varphi_k(t))$ via 4th order Runge-Kutta.

We then subject the flow lines and the circle (which is also a flow line) to a Kármán-Trefftz transform $= \frac{(z+1)^n - (z-1)^n}{(z+1)^n + (z-1)^n}, n \in \mathbb{R}, n \leq 2$. This transforms the circle into an airfoil-like shape with an angle $\beta = 2\pi - n\pi$ at the trailing edge $z=1$. For $n=2$, the Kármán-Trefftz transform is the same as the Joukowsky transform, which yields a cusp ($\beta = 0$) at the trailing edge. The

Kutta condition requires that the velocity be finite at the trailing edge. This can be guaranteed by choosing $\Gamma = 4\pi V_\infty[(1 - Re(c))\sin(\alpha) + Im(c)\cos(\alpha)]$.

For the mathematical details, see Jack Moran, *Theoretical and Computational Aerodynamics*, Wiley, 1984, chapter 4.9, and the Wikipedia article "Joukowsky transform".

The user can change the center c of the circle; the radius a is then adapted such that the circle still passes through the point (1,0), which is transformed into a cusp or an angle (the trailing edge of the airfoil) by the Kármán-Trefftz transform. The circulation Γ is also adapted automatically.

Controls:

arrow keys	move the center of the circle
[+],[-]	increase or decrease α, the angle of attack
[*],[/]	increase or decrease β, the angle at the trailing edge of the airfoil
Enter, [t]	toggle between the circle and its transform
Esc	show circle